# SERIES REVERSION/INVERSION OF LAMBERT'S TIME FUNCTION

THESIS

James Dana Thorne
Captain, USAF

AFIT/GA/ENY/89D-6

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY
AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

# REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704-0188

| | |
|---|---|
| 1a. REPORT SECURITY CLASSIFICATION<br>UNCLASSIFIED | 1b. RESTRICTIVE MARKINGS |
| 2a. SECURITY CLASSIFICATION AUTHORITY | 3. DISTRIBUTION/AVAILABILITY OF REPORT<br>Approved for public release;<br>distribution unlimited |
| 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE | |
| 4. PERFORMING ORGANIZATION REPORT NUMBER(S)<br>AFIT/GA/ENY/89D-6 | 5. MONITORING ORGANIZATION REPORT NUMBER(S) |
| 6a. NAME OF PERFORMING ORGANIZATION<br>School of Engineering<br>6b. OFFICE SYMBOL (If applicable)<br>AFIT/EN | 7a. NAME OF MONITORING ORGANIZATION |
| 6c. ADDRESS (City, State, and ZIP Code)<br>Air Force Institute of Technology<br>Wright-Patterson AFB OH 45433-6583 | 7b. ADDRESS (City, State, and ZIP Code) |
| 8a. NAME OF FUNDING/SPONSORING ORGANIZATION<br>8b. OFFICE SYMBOL (If applicable) | 9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER |
| 8c. ADDRESS (City, State, and ZIP Code) | 10. SOURCE OF FUNDING NUMBERS<br>PROGRAM ELEMENT NO. \| PROJECT NO \| TASK NO \| WORK UNIT ACCESSION NO |

11. TITLE (Include Security Classification)
SERIES REVERSION/INVERSION OF LAMBERT'S TIME FUNCTION

12. PERSONAL AUTHOR(S)
James D. Thorne, B.S., Capt, USAF

| 13a. TYPE OF REPORT | 13b. TIME COVERED | 14. DATE OF REPORT (Year, Month, Day) | 15. PAGE COUNT |
|---|---|---|---|
| MS Thesis | FROM ______ TO ______ | 1989 December | 50 |

16. SUPPLEMENTARY NOTATION

| 17. COSATI CODES | | | 18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) |
|---|---|---|---|
| FIELD | GROUP | SUB-GROUP | Celestial Mechanics |
| 03 | 03 | | |
| | | | |

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Thesis Advisor: Rodney D. Bain, Capt, USAF
Assistant Professor
Department of Aeronautics and Astronautics

| | |
|---|---|
| 20. DISTRIBUTION/AVAILABILITY OF ABSTRACT<br>☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS | 21. ABSTRACT SECURITY CLASSIFICATION<br>UNCLASSIFIED |
| 22a. NAME OF RESPONSIBLE INDIVIDUAL<br>Rodney D. Bain, Capt, USAF, Asst. Professor | 22b. TELEPHONE (Include Area Code)<br>(513) 255-3633<br>22c. OFFICE SYMBOL<br>ENY |

DD Form 1473, JUN 86 Previous editions are obsolete

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

The time of flight of a two-body orbit may be determined by integrating the radial velocity equation for a conic section. The resulting expression is sometimes called Lambert's Time Function, which depends on the gravitational constant, two position vectors, and the semi-major axis of the conic flight path. For mission planning purposes, it is often more desirable to know the semi-major axis as a function of time, rather than the reverse. Normally, a root finding technique such as Newton-Raphson is employed to find the value of a characteristic orbital parameter which matches a given time of flight. Alternatively, Lambert's Time Function may be expanded a power series involving the inverse semi-major axis. The expression for semi-major axis is then determined through series reversion and inversion of the resulting series. A simplified method of obtaining the series coefficients is given, as well as a numerical study of convergence properties.

(1)

AFIT/GA/ENY/89D-6

SERIES REVERSION/INVERSION OF
LAMBERT'S TIME FUNCTION

THESIS

James Dana Thorne
Captain, USAF

AFIT/GA/ENY/89D-6

AFIT/GA/ENY/89D-6

# SERIES REVERSION/INVERSION OF LAMBERT'S TIME FUNCTION

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

Requirements for the Degree of

Master of Science in Astronautical Engineering

James Dana Thorne, B.S.

Captain, USAF

December, 1989

*Acknowledgments*

I would like to acknowledge the contributions of the following individuals toward the preparation of this thesis:

Captain R. D. Bain, USAF for the introduction of many of the mathematical references used in the derivations, as well as the time and patience required of an advisor.

Lieutenant Colonel R. Bagley, USAF, for timely encouragement and a clear view of the "big picture."

Dr. W. F. Wiesel, for providing an opportunity to test the method presented herein on simulated Earth-Mars transfers.

My wife, Andrea D. Thorne for technical advice on computer programming techniques, as well as patience and consideration during the degree program.

The members of the 1988-1989 AFIT Astronautical Engineering student section for their friendship and support.

James Dana Thorne

*Table of Contents*

## *List of Figures*

## *List of Tables*

## *List of Symbols*



Shouldn't this be a nomenclature section?

*Abstract*

The time of flight of a two-body orbit may be determined by integrating the radial velocity equation for a conic section. The resulting expression is sometimes called Lambert's Time Function, which depends on the gravitational constant, two position vectors, and the semi-major axis of the conic flight path. For mission planning purposes, it is often desirable to know the semi-major axis as a function of time, rather than the reverse. Normally, a root finding technique such as Newton-Raphson is employed to find the value of a characteristic orbital parameter which matches a given time of flight. Alternatively, Lambert's Time Function may be expanded as a power series involving the inverse semi-major axis. The expression for semi-major axis is then determined through series reversion and inversion of the resulting series. A simplified method of obtaining the series coefficients is given, as well as a numerical study of convergence properties.

# SERIES REVERSION/INVERSION OF LAMBERT'S TIME FUNCTION

## *I. Introduction*

The two-position, time of flight problem in orbital mechanics has historically been the subject of many investigations. Although a closed form solution is available through integration of the polar velocity equation for a conic section, the resulting expression is transcendental in the orbital parameters, making it difficult to solve for them. In order to match the orbital parameters to a particular flight time, root finding techniques are normally used. Many different formulations of the problem have been developed to minimize the convergence time of these root finding methods, as well as to generalize the problem to avoid case dependent equations. In each of these methods, an initial value is required. Depending on the method used, convergence may not be achieved at all if the initial value is too different from the correct value. In the classical Gauss method (4: p. 188-197 ), for example, the algorithm fails to converge for transfer angles larger than roughly 70°. Although other methods have been developed to converge for larger angles, they may be deficient for small angles. It would be desirable, therefore, to develop a solution that does not require an initial value and is not dependent on transfer angle for convergence properties.

The method presented is a solution of the closed form time of flight equation for semi-major axis as a function of time. It will be shown that this function may be used to directly calculate the semi-major axis of an orbit without the utilization of a root finding technique.

# II. Analytical Development

## *The Two-Body Time of Flight Problem*

The two-body time of flight problem may be stated as follows: Given two position vectors from the gravitational body to the orbiting body and a time of flight between the positions, determine the semi-major axis of the conic trajectory followed by the body.

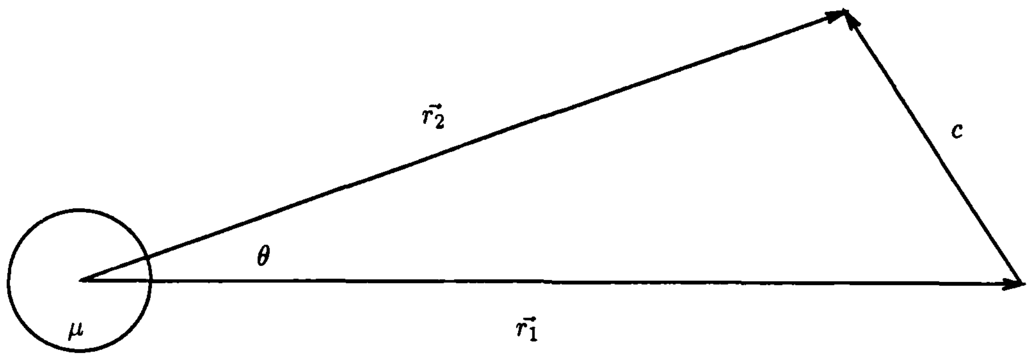


Figure 1. Geometry of two-body problem

In the problem statement above it is assumed that the Newtonian gravitational constant is known, and both bodies behave as point masses.

*Lambert's Time Function*

The purpose of this section is to derive Lambert's Time Function in a closed, transcendental form. Once known, it can then be expressed as a power series allowing for later reversion and inversion to solve for semi-major axis.

If the position of the orbiting body is expressed in polar coordinates, the *energy integral* may be written:

$$\left(\frac{dr}{dt}\right)^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{1}$$

where $r$ is the radial distance between the two bodies, $\mu$ is the gravitational constant, $a$ is the semi-major axis of the conic section, and $t$ is the time. Equation (1) may be rearranged as follows:

$$dt = \frac{r\,dr}{\sqrt{\mu}\sqrt{2r - (r^2/a)}} \tag{2}$$

The integral takes the form (3: p. 70-75 ):

$$t = \int_{s-c}^{s} \frac{r\,dr}{\sqrt{\mu}\sqrt{2r - (r^2/a)}} \tag{3}$$

where $s = (r_1 + r_2 + c)/2$, and $c =$ chord. ( see figure 1.)

The integration may be simplified by introducing a change of variable:

$$r = a(1 - \cos\phi) \tag{4}$$

To find the new limits of integration:

$$r_1 = s - c = a(1 - \cos\phi) \tag{5}$$

$$\phi = \cos^{-1}(1 - \frac{s-c}{a}) \tag{6}$$

$$\sin(\phi/2) = \sqrt{\frac{1}{2}(1 - \cos^{-1}(\cos(1 - \frac{s-c}{a})))} \tag{7}$$

$$\phi_l = 2\sin^{-1}\sqrt{\frac{s-c}{2a}} = \alpha \tag{8}$$

Similarly, the upper limit of integration becomes:

$$\phi_u = 2\sin^{-1}\sqrt{\frac{s}{2a}} = \beta \tag{9}$$

Letting $\alpha$ and $\beta$ be the lower and upper limits, respectively, the integral becomes:

$$t = \frac{1}{\sqrt{\mu}} \int_\alpha^\beta \frac{a(1-\cos\phi)a\sin\phi\, d\phi}{\sqrt{2a(1-\cos\phi) - a(1-\cos\phi)^2}} \tag{10}$$

which simplifies to:

$$t = \sqrt{\frac{a^3}{\mu}} \int_\alpha^\beta (1-\cos\phi)\, d\phi \tag{11}$$

Performing the quadrature yields:

$$t = \sqrt{\frac{a^3}{\mu}}[(\alpha - \sin\alpha) - (\beta - \sin\beta)] \tag{12}$$

This is Lambert's Time Function for an elliptic trajectory with a transfer angle less than $\pi$, and a flight time less than the minimum energy transfer time. A minimum energy transfer is an elliptical transfer where the semi-major axis is equal to one half of the semi-perimeter, $a = s/2$.

*Series Representation of Time of Flight*

In order to accomplish series reversion and inversion, the time of flight must first be expressed as a single power series. Substitution of the definitions of $\alpha$ and $\beta$ yields:

$$
\begin{aligned}
t &= \sqrt{\frac{a^3}{\mu}}\left[\left(2\sin^{-1}\sqrt{\frac{s}{2a}} - \sin\left(2\sin^{-1}\sqrt{\frac{s}{2a}}\right)\right)\right. \\
&\quad \left. - \left(2\sin^{-1}\sqrt{\frac{s-c}{2a}} - \sin\left(2\sin^{-1}\sqrt{\frac{s-c}{2a}}\right)\right)\right] \\
&= \sqrt{\frac{a^3}{\mu}}\left[\left(2\sin^{-1}\sqrt{\frac{s}{2a}} - 2\sqrt{\frac{s}{2a}}\cos\left(\sin^{-1}\sqrt{\frac{s}{2a}}\right)\right)\right. \\
&\quad \left. - \left(2\sin^{-1}\sqrt{\frac{s-c}{2a}} - 2\sqrt{\frac{s-c}{2a}}\cos\left(\sin^{-1}\sqrt{\frac{s-c}{2a}}\right)\right)\right] \\
&= \sqrt{\frac{a^3}{\mu}}\left[\left(2\sin^{-1}\sqrt{\frac{s}{2a}} - 2\sqrt{\frac{s}{2a}}\sqrt{1-\frac{s}{2a}}\right)\right. \\
&\quad \left. - \left(2\sin^{-1}\sqrt{\frac{s-c}{2a}} - 2\sqrt{\frac{s-c}{2a}}\sqrt{1-\frac{s-c}{2a}}\right)\right] \qquad (13)
\end{aligned}
$$

The Hypergeometric Series definitions for $\sin^{-1} x$ and $\sqrt{1-x^2}$ are:

$$\sin^{-1} x = xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)$$

$$\sqrt{1-x^2} = F\left(\frac{-1}{2}, b; b; x^2\right)$$

Substituting into equation (13) produces:

$$
\begin{aligned}
t &= 2\sqrt{\frac{a^3}{\mu}}\left[\left(\sqrt{\frac{s}{2a}}F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{s}{2a}\right) - \sqrt{\frac{s}{2a}}F\left(\frac{-1}{2}, b; b; \frac{s}{2a}\right)\right)\right. \\
&\quad \left. - \left(\sqrt{\frac{s-c}{2a}}F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{s-c}{2a}\right) - \sqrt{\frac{s-c}{2a}}F\left(\frac{-1}{2}, b; b; \frac{s-c}{2a}\right)\right)\right]
\end{aligned}
$$

$$= \sqrt{\frac{s^3}{2\mu}}\left(\frac{s}{2a}\right)^{-1}\left[F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{s}{2a}\right) - F\left(\frac{-1}{2},b;b;\frac{s}{2a}\right)\right] -$$
$$\sqrt{\frac{(s-c)^3}{2\mu}}\left(\frac{s-c}{2a}\right)^{-1}\left[F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{s-c}{2a}\right) - F\left(\frac{-1}{2},b;b;\frac{s-c}{2a}\right)\right] \quad (14)$$

The hypergeometric functions may be expressed as infinite series, allowing manipulation of individual terms to form a single, combined series. Representing the hypergeometric functions with Pochhammer notation yields:

$$t = \sqrt{\frac{s^3}{2\mu}}\left(\frac{s}{2a}\right)^{-1}\sum_{n=0}^{\infty}\left[\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} - \left(\frac{-1}{2}\right)_n\right]\frac{1}{n!}\left(\frac{s}{2a}\right)^n -$$
$$\sqrt{\frac{(s-c)^3}{2\mu}}\left(\frac{s-c}{2a}\right)^{-1}\sum_{n=0}^{\infty}\left[\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} - \left(\frac{-1}{2}\right)_n\right]\frac{1}{n!}\left(\frac{s-c}{2a}\right)^n$$
$$= \sqrt{\frac{s^3}{2\mu}}\sum_{n=0}^{\infty}\left[\left(\frac{1}{2}\right)_n\frac{n}{(n^2-1/4)}\right]\frac{1}{n!}\left(\frac{s}{2a}\right)^{n-1} -$$
$$\sqrt{\frac{(s-c)^3}{2\mu}}\sum_{n=0}^{\infty}\left[\left(\frac{1}{2}\right)_n\frac{n}{(n^2-1/4)}\right]\frac{1}{n!}\left(\frac{s-c}{2a}\right)^{n-1} \quad (15)$$

The first terms in both series of equation (15) are zero. Changing the indicies by $m = n-1$, $n = m+1$ produces:

$$t = \sqrt{\frac{s^3}{2\mu}}\sum_{m=0}^{\infty}\left[\left(\frac{1}{2}\right)_{m+1}\frac{m+1}{((m+1)^2-1/4)}\right]\frac{1}{(m+1)!}\left(\frac{s}{2a}\right)^m -$$
$$\sqrt{\frac{(s-c)^3}{2\mu}}\sum_{m=0}^{\infty}\left[\left(\frac{1}{2}\right)_{m+1}\frac{m+1}{((m+1)^2-1/4)}\right]\frac{1}{(m+1)!}\left(\frac{s-c}{2a}\right)^m$$
$$= \frac{2}{3}\sqrt{\frac{s^3}{2\mu}}\sum_{m=0}^{\infty}\left[\frac{\left(\frac{3}{2}\right)_m}{(4/3)(m+1)^2-1/3}\right]\frac{1}{(m)!}\left(\frac{s}{2a}\right)^m -$$
$$\frac{2}{3}\sqrt{\frac{(s-c)^3}{2\mu}}\sum_{m=0}^{\infty}\left[\frac{\left(\frac{3}{2}\right)_m}{(4/3)(m+1)^2-1/3}\right]\frac{1}{(m)!}\left(\frac{s-c}{2a}\right)^m \quad (16)$$

In order to produce the hypergeometric parameters (1: p. 276-277), one must form the ratio $\frac{u_{m+1}}{u_m}$ where $u_m$ is the $m^{th}$ term of the infinite series. This gives the hypergeometric parameters $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2})$ by inspection:

$$(u_{m+1}/u_m) = \left[\frac{(m+1/2)(m+3/2)}{(m+5/2)}\right] \frac{1}{m+1} \left(\frac{s}{2a}\right) \tag{17}$$

Substitution into equation (16) yields:

$$\begin{aligned}
t &= \frac{2}{3}\sqrt{\frac{s^3}{2\mu}} F\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; \left(\frac{s}{2a}\right)\right) - \frac{2}{3}\sqrt{\frac{(s-c)^3}{2\mu}} F\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; \left(\frac{s-c}{2a}\right)\right) \\
&= \frac{2}{3}\sqrt{\frac{s^3}{2\mu}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n n!} \left(\frac{s}{2a}\right)^n - \sqrt{\frac{(s-c)^3}{2\mu}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n n!} \left(\frac{s-c}{2a}\right)^n \\
&= \frac{2}{3}\sqrt{\frac{s^3}{2\mu}} \sum_{n=0}^{\infty} \left(1 - \left(\frac{s-c}{s}\right)^{n+3/2}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n n!} \left(\frac{s}{2a}\right)^n \\
t &= \frac{2}{3}\sqrt{\frac{s^3}{2\mu}} \left(1 - \left(\frac{s-c}{s}\right)^{3/2}\right) \\
&\quad + \frac{2}{3}\sqrt{\frac{s^3}{2\mu}} \sum_{n=1}^{\infty} \left(1 - \left(\frac{s-c}{s}\right)^{n+3/2}\right) \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n n!} \left(\frac{s}{2a}\right)^n
\end{aligned} \tag{18}$$

The first term may be recognized as the Parabolic Transfer Time, $t_p$ , for the given problem geometry. ( see figure 1 ) The Parabolic Transfer Time is the flight time for an object traveling on a parabolic trajectory between two position vectors, where the origin of the position vectors is the focus of the parabola. The flight time of an object traveling on a hyperbolic trajectory will be less than the Parabolic Transfer Time, and the flight time on an elliptic trajectory will be greater. The semi-major axis of a parabola is infinite by definition, and is the limiting case between hyperbolas and ellipses. Let the quantity $\left(\frac{t}{t_p} - 1\right) = T$, a non-dimensional time parameter.

$$T = \left(\frac{t}{t_p} - 1\right) = \sum_{n=1}^{\infty} \frac{\left(1 - \left(\frac{s-c}{s}\right)^{n+3/2}\right)}{\left(1 - \left(\frac{s-c}{s}\right)^{3/2}\right)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n n!} \left(\frac{s}{2a}\right)^n \tag{19}$$

The gravitational constant $\mu$ is included only in the non-dimensional time parameter $T$. Therefore, the series coefficients are functions of geometric constants.

It may be shown that the time equation for the hyperbolic case differs only in the sign of the argument, i. e., $(\frac{-s}{2a})$, and the series coefficients are identical.

*Reversion/Inversion of Series*

The purpose of this section is to solve equation (19) for the semi-major axis, $a$. To this end, a series reversion will be accomplished followed by a binomial expansion of its inverse. In general, equation (19) may be expressed as follows:

$$T = A_1\left(\frac{s}{2a}\right) + A_2\left(\frac{s}{2a}\right)^2 + A_3\left(\frac{s}{2a}\right)^3 + ... \tag{20}$$

In a series reversion, the expression is rewritten (2: p. 316-317):

$$\left(\frac{s}{2a}\right) = A_1'T + A_2'T^2 + A_3'T^3 + ... \tag{21}$$

In order to determine the semi-major axis, the inverse must be found:

$$\begin{aligned}
\left(\frac{2a}{s}\right) &= (A_1'T + A_2'T^2 + A_3'T^3 + ...)^{-1} \\
&= \sum_{n=0}^{\infty}(-1)^n(A_1'T)^{(-1-n)}(A_2'T^2 + A_3'T^3 + ...)^n \\
&= (A_1'T)^{-1} - (A_1'T)^{-2}(A_2'T^2 + A_3'T^3 + ...) \\
&\quad +(A_1'T)^{-3}(A_2'T^2 + A_3'T^3 + ...)^2 - ...
\end{aligned} \tag{22}$$

Or more generally:

$$\left(\frac{2a}{s}\right) = B_1T^{-1} + B_2T^0 + B_3T^1 + B_4T^2 + ... \tag{23}$$

by equating powers of $T$.

*Determination of Series Coefficients*

Now that the semi-major axis has been expressed as a power series in T, a method must be developed to determine the unknown coefficients $B_i$ in terms of the known coefficients $A_i$. Equation (23) is repeated here for convenience:

$$\left(\frac{2a}{s}\right) = B_1 T^{-1} + B_2 T^0 + B_3 T^1 + B_4 T^2 + \ldots \tag{24}$$

Multiply by $\left(\frac{s}{2a}\right) T$ and substitute for $T$:

$$\begin{aligned} &A_1 \left(\tfrac{s}{2a}\right) + A_2 \left(\tfrac{s}{2a}\right)^2 + A_3 \left(\tfrac{s}{2a}\right)^3 + \ldots = \\ &\quad B_1 \left(\tfrac{s}{2a}\right) + B_2 \left(\tfrac{s}{2a}\right) \left\{A_1 \left(\tfrac{s}{2a}\right) + A_2 \left(\tfrac{s}{2a}\right)^2 + A_3 \left(\tfrac{s}{2a}\right)^3 + \ldots\right\} + \ldots \end{aligned} \tag{25}$$

Taking a derivative with respect to $\left(\frac{s}{2a}\right)$ yields:

$$\begin{aligned} &A_1 + 2A_2 \left(\tfrac{s}{2a}\right) + 3A_3 \left(\tfrac{s}{2a}\right)^2 + \ldots = \\ &\quad B_1 + B_2 \left\{A_1 \left(\tfrac{s}{2a}\right) + A_2 \left(\tfrac{s}{2a}\right)^2 + A_3 \left(\tfrac{s}{2a}\right)^3 + \ldots\right\} \\ &\quad + B_2 \left(\tfrac{s}{2a}\right) \left\{A_1 + 2A_2 \left(\tfrac{s}{2a}\right) + 3A_3 \left(\tfrac{s}{2a}\right)^2 + \ldots\right\} + \ldots \end{aligned} \tag{26}$$

Evaluate at $\left(\frac{s}{2a}\right) = 0$:

$$A_1 = B_1 \tag{27}$$

In an analogous fashion, successive derivatives evaluated at zero produce the following:

$$A_2 = A_1 B_2 \tag{28}$$

$$A_3 = A_2 B_2 + A_1^2 B_3 \tag{29}$$

The expressions become increasingly complex with each derivative.

In general, a matrix equation may be formed showing the relationships between $A_i$

and $B_i$:

$$\left\{\begin{array}{c} A_2 \\ A_3 \\ A_4 \\ . \\ A_i \end{array}\right\} = \left[\begin{array}{ccccc} A_1 & 0 & 0 & 0 & 0 \\ A_2 & A_1^2 & 0 & 0 & 0 \\ A_3 & 2A_1A_2 & A_1^3 & 0 & 0 \\ . & . & . & . & 0 \\ A_i & . & . & . & A_1^i \end{array}\right] \left\{\begin{array}{c} B_2 \\ B_3 \\ B_4 \\ . \\ B_i \end{array}\right\} = Q^{-1}\vec{B} \qquad (30)$$

Solving for the $B_i$:

$$\left\{\begin{array}{c} B_2 \\ B_3 \\ B_4 \\ . \\ B_i \end{array}\right\} = \left[\begin{array}{ccccc} A_1^{-1} & 0 & 0 & 0 & 0 \\ -A_2/A_1^3 & A_1^{-2} & 0 & 0 & 0 \\ (2A_2^2 - A_1A_3)/A_1^5 & -2A_2/A_1^4 & A_1^{-3} & 0 & 0 \\ . & . & . & . & 0 \\ . & . & . & . & A_1^{-i} \end{array}\right] \left\{\begin{array}{c} A_2 \\ A_3 \\ A_4 \\ . \\ A_i \end{array}\right\} = Q\vec{A} \qquad (31)$$

To summarize:

$$B_1 = A_1 \qquad (32)$$

$$B_2 = A_2/A_1 \qquad (33)$$

$$B_3 = (A_1A_3 - A_2^2)/A_1^3 \qquad (34)$$

$$B_4 = (A_1^2A_4 - 3A_1A_2A_3 + 2A_2^3)/A_1^5 \qquad (35)$$

It may be seen that the elements of the matrix in equation (31) follow a pattern. This allows the matrix to be formed without the necessity of taking numerous derivatives as was shown previously. As an aid to finding this pattern, the computer program MACSYMA, a trademark of Symbolics, Inc. (Project MAC's SYmbolic MAnipulation System) was used to generate numerous terms for inspection. The infinite series were represented by tenth order polynomials and then expanded by MACSYMA, which also evaluated ten derivatives in the manner of equation (26). This produced ten relationships between the $A_i$ and the $B_i$ which were arranged in matrix form. Finally, MACSYMA was used to symbolically invert a subset of the matrix allowing for the pattern to be recognized through inspection, trial and error. It should be emphasized that a symbolic manipulation program such as

MACSYMA may prove to be an invaluable tool in this type of analysis.

To form the matrix in equation (31), element $q_{11}$ is assumed to be $A_1^{-1}$. From this, all remaining elements may be formed. The process is illustrated by example:

To form element $q_{33}$, take the following dot product:

$$q_{33} = \{q_{21}, q_{11}\} \bullet \{q_{12}, q_{22}\} \tag{36}$$

$$= \left\{-A_2/A_1^3, A_1^{-1}\right\} \bullet \left\{0, A_1^{-2}\right\} = A_1^{-3} \tag{37}$$

To form element $q_{32}$, take the following dot product:

$$q_{32} = \{q_{21}, q_{11}\} \bullet \{q_{11}, q_{21}\} \tag{38}$$

$$= \left\{-A_2/A_1^3, A_1^{-1}\right\} \bullet \left\{A_1^{-1}, -A_2/A_1^3\right\} = -2A_2/A_1^4 \tag{39}$$

To form element $q_{31}$:

$$q_{31} = -\frac{1}{A_1}\{q_{32}, q_{33}, q_{34}, ...\} \bullet \{A_2, A_3, A_4, ...\} \tag{40}$$

$$= -\frac{1}{A_1}\left\{-2A_2^2/A_1^4 + A_3/A_1^3\right\} = (2A_2^2 - A_1A_3)/A_1^5 \tag{41}$$

All of the elements of the third row are now determined, and the fourth row may be found beginning with $q_{44}$ and ending with $q_{41}$. The first element of each new row must be calculated last, since all of the remaining row elements are required. To generalize the procedures, the following steps should be performed:

1. $q_{11} = A_1^{-1}$
2. $q_{ij} = (q_{(i-1,1)}, q_{(i-2,1)}, ...q_{11}) \bullet (q_{(1,j-1)}, q_{(2,j-1)}, ...q_{(i-1,j-1)}),\ 2 < j <= i$
3. $q_{i1} = (-A_1^{-1})\left\{(q_{i2}, q_{i3}, ...q_{i,i}) \bullet (-A_1^{-1})(A_2, A_3, ...A_i)\right\}$
4. $B_i = (q_{i1}, q_{i2}, ..., q_{ii}) \bullet (A_2, A_3, ..., A_{i+1})$

Repeat steps 2, 3, and 4 for each additional term desired. ( see Appendix C, "Form Q Matrix" ) Because the $A_i$ are known from equation (19) they may be numerically determined beforehand and used to calculate the $B_i$ coefficients.

In the figure below, $r_1 = r_2 = k = 1$, and $\theta = 180^o$. Recall $T = 0$ corresponds to a parabolic transfer.

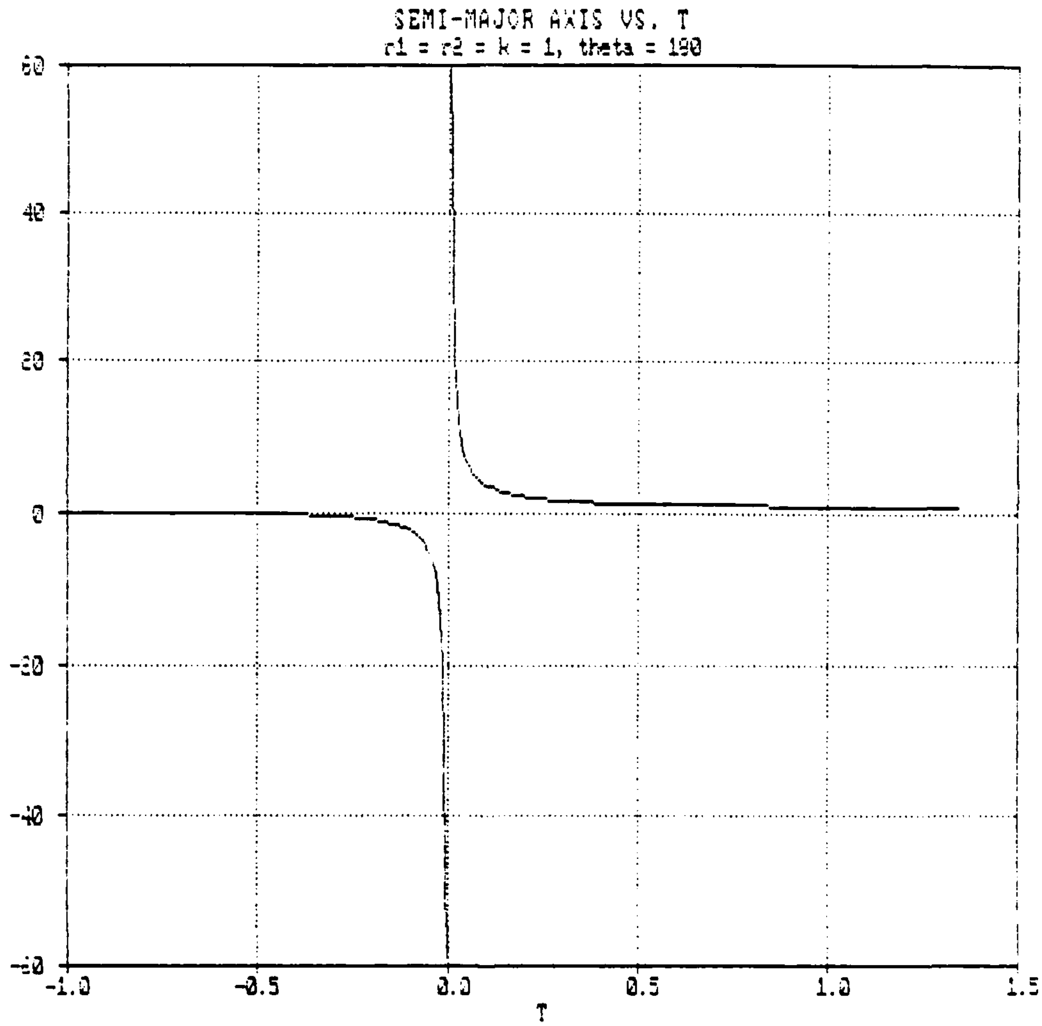


Figure 2. Semi-major axis vs. T for general 180° case.

# III. Numerical Investigation and Results

## General Case

The general case considered was composed of the following orbital geometry:

- magnitudes of position vectors: $r_1 = r_2 = 1$
- gravitational constant: $\mu = 1$
- range of transfer angle: $0^o \leq \theta \leq 360^o$, $15^o$ increments
- range of flight time: $0 \leq t \leq t_{me}$

The following data consists of values of semi-major axis calculated for each $15^o$ of transfer angle using partial sums of the series. Values are tabulated for terms $i = 1, 2, 3$ and $i = 21, 22, 23$. The last column is the value of $T$ obtained by using the first 23 terms to find the semi-major axis, then by employing Lambert's Time Function. These values of $T$ may be used as an accuracy check against the desired value of $T$ shown in the caption of each table. Taking an example row of data, for $T = -0.9$ and $\theta = 30^o$, the semi-major axis value using one term in the series is -0.28090, for two terms is -0.10653, and for terms 21, 22, and 23 remains a constant -0.00504. At this point the semi-major axis seems to have converged to the correct value, since application of Lambert's Time Function yields $T = -0.9$ as the last entry in the data row.

The accuracy of the series is plotted against $T$ and $i$ in Appendix B for transfer angles of $90^o, 180^o, 270^o$, and $360^o$. The vertical axis represents the number of significant figures in the semi-major axis value. The four plots have a very similar appearance, which would seem to indicate a general insensitivity of accuracy with respect to the transfer angle. Plots were constructed for every $10^o$, but because of the similarities, only four plots are presented.

Table 1. Values of semi-major axis for the general case, $T = -0.9$

| $\theta$ | 1 | 2 | 3 | | 21 | 22 | 23 | $T$ |
|---|---|---|---|---|---|---|---|---|
| 15.0 | −0.27857 | 0.09962 | 0.03942 | ... | −0.00504 | −0.00504 | −0.00504 | −0.9000 |
| 30.0 | −0.28090 | 0.10653 | 0.03527 | ... | −0.00502 | −0.00502 | −0.00502 | −0.9000 |
| 45.0 | −0.28468 | 0.11718 | 0.02979 | ... | −0.00499 | −0.00499 | −0.00499 | −0.9000 |
| 60.0 | −0.28972 | 0.13043 | 0.02450 | ... | −0.00494 | −0.00494 | −0.00494 | −0.9000 |
| 75.0 | −0.29578 | 0.14494 | 0.02058 | ... | −0.00489 | −0.00489 | −0.00489 | −0.9000 |
| 90.0 | −0.30255 | 0.15939 | 0.01853 | ... | −0.00483 | −0.00483 | −0.00483 | −0.9000 |
| 105.0 | −0.30966 | 0.17263 | 0.01819 | ... | −0.00476 | −0.00476 | −0.00476 | −0.9000 |
| 120.0 | −0.31667 | 0.18380 | 0.01894 | ... | −0.00470 | −0.00470 | −0.00469 | −0.9000 |
| 135.0 | −0.32309 | 0.19239 | 0.02007 | ... | −0.00464 | −0.00464 | −0.00464 | −0.8999 |
| 150.0 | −0.32839 | 0.19822 | 0.02102 | ... | −0.00458 | −0.00458 | −0.00458 | −0.9001 |
| 165.0 | −0.33200 | 0.20142 | 0.02155 | ... | −0.00456 | −0.00456 | −0.00456 | −0.9000 |
| 180.0 | −0.33333 | 0.20238 | 0.02168 | ... | −0.00456 | −0.00456 | −0.00456 | −0.9000 |
| 195.0 | −0.33181 | 0.20161 | 0.02163 | ... | −0.00452 | −0.00452 | −0.00453 | −0.9000 |
| 210.0 | −0.32692 | 0.19965 | 0.02160 | ... | −0.00439 | −0.00439 | −0.00439 | −0.9000 |
| 225.0 | −0.31824 | 0.19693 | 0.02161 | ... | −0.00415 | −0.00415 | −0.00415 | −0.9002 |
| 240.0 | −0.30556 | 0.19363 | 0.02130 | ... | −0.00390 | −0.00391 | −0.00391 | −0.8997 |
| 255.0 | −0.28894 | 0.18952 | 0.01986 | ... | −0.00356 | −0.00354 | −0.00354 | −0.9005 |
| 270.0 | −0.26888 | 0.18382 | 0.01617 | ... | −0.00335 | −0.00335 | −0.00332 | −0.8995 |
| 285.0 | −0.24640 | 0.17528 | 0.00944 | ... | −0.00287 | −0.00301 | −0.00311 | −0.8982 |
| 300.0 | −0.22310 | 0.16254 | 0.00059 | ... | −0.00249 | −0.00261 | −0.00288 | −0.8977 |
| 315.0 | −0.20114 | 0.14513 | −0.00620 | ... | −0.00221 | −0.00250 | −0.00283 | −0.8948 |
| 330.0 | −0.18296 | 0.12509 | −0.00486 | ... | −0.00233 | −0.00260 | −0.00232 | −0.9017 |
| 345.0 | −0.17090 | 0.10804 | 0.00470 | ... | −0.00233 | −0.00229 | −0.00233 | −0.8995 |
| 360.0 | −0.16667 | 0.10119 | 0.01084 | ... | −0.00228 | −0.00228 | −0.00228 | −0.9000 |

Table 2. Values of semi-major axis for the general case, $T = -0.4$

| $\theta$ | 1 | 2 | 3 | | 21 | 22 | 23 | $T$ |
|---|---|---|---|---|---|---|---|---|
| 15.0 | −0.62678 | −0.24859 | −0.27535 | ... | −0.28128 | −0.28128 | −0.28128 | −0.4000 |
| 30.0 | −0.63203 | −0.24460 | −0.27627 | ... | −0.28140 | −0.28140 | −0.28140 | −0.4000 |
| 45.0 | −0.64053 | −0.23867 | −0.27751 | ... | −0.28167 | −0.28167 | −0.28167 | −0.4000 |
| 60.0 | −0.65187 | −0.23172 | −0.27880 | ... | −0.28217 | −0.28217 | −0.28217 | −0.4000 |
| 75.0 | −0.66550 | −0.22479 | −0.28006 | ... | −0.28303 | −0.28303 | −0.28303 | −0.4000 |
| 90.0 | −0.68074 | −0.21880 | −0.28140 | ... | −0.28437 | −0.28437 | −0.28437 | −0.4000 |
| 105.0 | −0.69673 | −0.21444 | −0.28308 | ... | −0.28629 | −0.28629 | −0.28629 | −0.4000 |
| 120.0 | −0.71250 | −0.21203 | −0.28530 | ... | −0.28883 | −0.28883 | −0.28883 | −0.4000 |
| 135.0 | −0.72695 | −0.21147 | −0.28806 | ... | −0.29184 | −0.29184 | −0.29184 | −0.4000 |
| 150.0 | −0.73888 | −0.21227 | −0.29102 | ... | −0.29495 | −0.29495 | −0.29495 | −0.4000 |
| 165.0 | −0.74700 | −0.21358 | −0.29352 | ... | −0.29752 | −0.29752 | −0.29752 | −0.4000 |
| 180.0 | −0.75000 | −0.21429 | −0.29460 | ... | −0.29862 | −0.29862 | −0.29862 | −0.4000 |
| 195.0 | −0.74658 | −0.21316 | −0.29315 | ... | −0.29716 | −0.29716 | −0.29716 | −0.4000 |
| 210.0 | −0.73557 | −0.20901 | −0.28814 | ... | −0.29211 | −0.29211 | −0.29211 | −0.4000 |
| 225.0 | −0.71604 | −0.20087 | −0.27879 | ... | −0.28276 | −0.28276 | −0.28276 | −0.4000 |
| 240.0 | −0.68750 | −0.18831 | −0.26490 | ... | −0.26889 | −0.26889 | −0.26889 | −0.4000 |
| 255.0 | −0.65011 | −0.17165 | −0.24706 | ... | −0.25104 | −0.25104 | −0.25104 | −0.4000 |
| 270.0 | −0.60498 | −0.15228 | −0.22679 | ... | −0.23052 | −0.23052 | −0.23052 | −0.4000 |
| 285.0 | −0.55439 | −0.13272 | −0.20642 | ... | −0.20921 | −0.20921 | −0.20921 | −0.4000 |
| 300.0 | −0.50198 | −0.11634 | −0.18831 | ... | −0.18920 | −0.18920 | −0.18920 | −0.4000 |
| 315.0 | −0.45256 | −0.10629 | −0.17355 | ... | −0.17228 | −0.17228 | −0.17228 | −0.4000 |
| 330.0 | −0.41165 | −0.10360 | −0.16136 | ... | −0.15964 | −0.15964 | −0.15964 | −0.4000 |
| 345.0 | −0.38452 | −0.10558 | −0.15151 | ... | −0.15190 | −0.15190 | −0.15190 | −0.4000 |
| 360.0 | −0.37500 | −0.10714 | −0.14730 | ... | −0.14931 | −0.14931 | −0.14931 | −0.4000 |

Table 3. Values of semi-major axis for the general case, $T = 0.1$

| $\theta$ | 1 | 2 | 3 | | 21 | 22 | 23 | $T$ |
|---|---|---|---|---|---|---|---|---|
| 15.0 | 2.50711 | 2.88530 | 2.89199 | ... | 2.89171 | 2.89171 | 2.89171 | 0.1000 |
| 30.0 | 2.52813 | 2.91557 | 2.92349 | ... | 2.92326 | 2.92326 | 2.92326 | 0.1000 |
| 45.0 | 2.56211 | 2.96397 | 2.97368 | ... | 2.97351 | 2.97351 | 2.97351 | 0.1000 |
| 60.0 | 2.60747 | 3.02762 | 3.03939 | ... | 3.03924 | 3.03924 | 3.03924 | 0.1000 |
| 75.0 | 2.66201 | 3.10273 | 3.11655 | ... | 3.11641 | 3.11641 | 3.11641 | 0.1000 |
| 90.0 | 2.72295 | 3.18489 | 3.20054 | ... | 3.20039 | 3.20039 | 3.20039 | 0.1000 |
| 105.0 | 2.78692 | 3.26920 | 3.28636 | ... | 3.28619 | 3.28619 | 3.28619 | 0.1000 |
| 120.0 | 2.85000 | 3.35047 | 3.36879 | ... | 3.36860 | 3.36860 | 3.36860 | 0.1000 |
| 135.0 | 2.90781 | 3.42328 | 3.44243 | ... | 3.44223 | 3.44223 | 3.44223 | 0.1000 |
| 150.0 | 2.95552 | 3.48212 | 3.50181 | ... | 3.50160 | 3.50160 | 3.50160 | 0.1000 |
| 165.0 | 2.98801 | 3.52143 | 3.54141 | ... | 3.54120 | 3.54120 | 3.54120 | 0.1000 |
| 180.0 | 3.00000 | 3.53571 | 3.55579 | ... | 3.55558 | 3.55558 | 3.55558 | 0.1000 |
| 195.0 | 2.98633 | 3.51975 | 3.53975 | ... | 3.53954 | 3.53954 | 3.53954 | 0.1000 |
| 210.0 | 2.94229 | 3.46886 | 3.48864 | ... | 3.48843 | 3.48843 | 3.48843 | 0.1000 |
| 225.0 | 2.86418 | 3.37935 | 3.39883 | ... | 3.39862 | 3.39862 | 3.39862 | 0.1000 |
| 240.0 | 2.75000 | 3.24919 | 3.26834 | ... | 3.26813 | 3.26813 | 3.26813 | 0.1000 |
| 255.0 | 2.60044 | 3.07889 | 3.09774 | ... | 3.09753 | 3.09753 | 3.09753 | 0.1000 |
| 270.0 | 2.41991 | 2.87260 | 2.89123 | ... | 2.89102 | 2.89102 | 2.89102 | 0.1000 |
| 285.0 | 2.21757 | 2.63925 | 2.65767 | ... | 2.65749 | 2.65749 | 2.65749 | 0.1000 |
| 300.0 | 2.00791 | 2.39355 | 2.41155 | ... | 2.41145 | 2.41145 | 2.41145 | 0.1000 |
| 315.0 | 1.81024 | 2.15650 | 2.17332 | ... | 2.17337 | 2.17337 | 2.17337 | 0.1000 |
| 330.0 | 1.64660 | 1.95465 | 1.96909 | ... | 1.96925 | 1.96925 | 1.96925 | 0.1000 |
| 345.0 | 1.53806 | 1.81700 | 1.82848 | ... | 1.82852 | 1.82852 | 1.82852 | 0.1000 |
| 360.0 | 1.50000 | 1.76786 | 1.77790 | ... | 1.77779 | 1.77779 | 1.77779 | 0.1000 |

Table 4. Values of semi-major axis for the general case, $T = 0.6$

| $\theta$ | 1 | 2 | 3 | | 21 | 22 | 23 | $T$ |
|---|---|---|---|---|---|---|---|---|
| 15.0 | 0.41785 | 0.79604 | 0.83617 | ... | 0.82821 | 0.82821 | 0.82821 | 0.6000 |
| 30.0 | 0.42136 | 0.80879 | 0.85630 | ... | 0.84994 | 0.84994 | 0.84994 | 0.6000 |
| 45.0 | 0.42702 | 0.82888 | 0.88714 | ... | 0.88229 | 0.88229 | 0.88229 | 0.6000 |
| 60.0 | 0.43458 | 0.85472 | 0.92534 | ... | 0.92125 | 0.92125 | 0.92125 | 0.6000 |
| 75.0 | 0.44367 | 0.88438 | 0.96729 | ... | 0.96315 | 0.96315 | 0.96315 | 0.6000 |
| 90.0 | 0.45383 | 0.91576 | 1.00966 | ... | 1.00498 | 1.00498 | 1.00498 | 0.6000 |
| 105.0 | 0.46449 | 0.94677 | 1.04973 | ... | 1.04438 | 1.04438 | 1.04438 | 0.6000 |
| 120.0 | 0.47500 | 0.97547 | 1.08538 | ... | 1.07946 | 1.07946 | 1.07946 | 0.6000 |
| 135.0 | 0.48463 | 1.00011 | 1.11500 | ... | 1.10868 | 1.10868 | 1.10868 | 0.6000 |
| 150.0 | 0.49259 | 1.01919 | 1.13732 | ... | 1.13078 | 1.13078 | 1.13078 | 0.6000 |
| 165.0 | 0.49800 | 1.03142 | 1.15134 | ... | 1.14468 | 1.14468 | 1.14468 | 0.6000 |
| 180.0 | 0.50000 | 1.03571 | 1.15618 | ... | 1.14948 | 1.14948 | 1.14948 | 0.6000 |
| 195.0 | 0.49772 | 1.03114 | 1.15113 | ... | 1.14446 | 1.14446 | 1.14446 | 0.6000 |
| 210.0 | 0.49038 | 1.01695 | 1.13564 | ... | 1.12903 | 1.12903 | 1.12903 | 0.6000 |
| 225.0 | 0.47736 | 0.99253 | 1.10941 | ... | 1.10285 | 1.10285 | 1.10285 | 0.6000 |
| 240.0 | 0.45833 | 0.95752 | 1.07241 | ... | 1.06583 | 1.06583 | 1.06583 | 0.6000 |
| 255.0 | 0.43341 | 0.91186 | 1.02497 | ... | 1.01829 | 1.01829 | 1.01829 | 0.6000 |
| 270.0 | 0.40332 | 0.85602 | 0.96778 | ... | 0.96103 | 0.96103 | 0.96103 | 0.6000 |
| 285.0 | 0.36960 | 0.79127 | 0.90183 | ... | 0.89548 | 0.89548 | 0.89548 | 0.6000 |
| 300.0 | 0.33465 | 0.72029 | 0.82825 | ... | 0.82380 | 0.82380 | 0.82380 | 0.6000 |
| 315.0 | 0.30171 | 0.64797 | 0.74886 | ... | 0.74893 | 0.74893 | 0.74893 | 0.6000 |
| 330.0 | 0.27443 | 0.58248 | 0.66912 | ... | 0.67482 | 0.67482 | 0.67482 | 0.6000 |
| 345.0 | 0.25634 | 0.53528 | 0.60417 | ... | 0.60857 | 0.60857 | 0.60857 | 0.6000 |
| 360.0 | 0.25000 | 0.51786 | 0.57809 | ... | 0.57474 | 0.57474 | 0.57474 | 0.6000 |

Table 5. Values of semi-major axis for the general case, $T = 1.1$

| $\theta$ | 1 | 2 | 3 | | 21 | 22 | 23 | $T$ |
|---|---|---|---|---|---|---|---|---|
| 15.0 | 0.22792 | 0.60611 | 0.67968 | ... | 0.65760 | 0.65760 | 0.65760 | 1.1000 |
| 30.0 | 0.22983 | 0.61726 | 0.70437 | ... | 0.68722 | 0.68722 | 0.68722 | 1.1000 |
| 45.0 | 0.23292 | 0.63478 | 0.74160 | ... | 0.72855 | 0.72855 | 0.72855 | 1.1000 |
| 60.0 | 0.23704 | 0.65719 | 0.78666 | ... | 0.77515 | 0.77515 | 0.77515 | 1.1000 |
| 75.0 | 0.24200 | 0.68272 | 0.83471 | ... | 0.82246 | 0.82246 | 0.82246 | 1.1000 |
| 90.0 | 0.24754 | 0.70948 | 0.88163 | ... | 0.86747 | 0.86748 | 0.86746 | 1.1001 |
| 105.0 | 0.25336 | 0.73564 | 0.92440 | ... | 0.90820 | 0.90812 | 0.90822 | 1.0994 |
| 120.0 | 0.25909 | 0.75956 | 0.96106 | ... | 0.94302 | 0.94325 | 0.94306 | 1.1010 |
| 135.0 | 0.26435 | 0.77983 | 0.99044 | ... | 0.97144 | 0.97134 | 0.97137 | 1.1000 |
| 150.0 | 0.26868 | 0.79529 | 1.01186 | ... | 0.99217 | 0.99205 | 0.99218 | 1.0992 |
| 165.0 | 0.27164 | 0.80506 | 1.02491 | ... | 1.00483 | 1.00480 | 1.00484 | 1.0997 |
| 180.0 | 0.27273 | 0.80844 | 1.02929 | ... | 1.00913 | 1.00909 | 1.00913 | 1.0997 |
| 195.0 | 0.27148 | 0.80491 | 1.02488 | ... | 1.00478 | 1.00475 | 1.00478 | 1.0998 |
| 210.0 | 0.26748 | 0.79405 | 1.01166 | ... | 0.99169 | 0.99175 | 0.99169 | 1.1004 |
| 225.0 | 0.26038 | 0.77555 | 0.98983 | ... | 0.97007 | 0.97000 | 0.97016 | 1.0986 |
| 240.0 | 0.25000 | 0.74919 | 0.95982 | ... | 0.94033 | 0.93992 | 0.94000 | 1.1007 |
| 255.0 | 0.23640 | 0.71486 | 0.92222 | ... | 0.90156 | 0.90239 | 0.90251 | 1.0945 |
| 270.0 | 0.21999 | 0.67269 | 0.87759 | ... | 0.85796 | 0.85796 | 0.85534 | 1.1485 |
| 285.0 | 0.20160 | 0.62327 | 0.82596 | ... | 0.80939 | 0.80155 | 0.80800 | 1.0538 |
| 360.0 | 0.13636 | 0.40422 | 0.51465 | ... | 0.50456 | 0.50454 | 0.50456 | 1.0997 |

Some transfer angle cases were omitted because $T = 1.1$ exceeded the minimum energy transfer time, $t_{me}$.

Table 6. Comparison of Gauss Method with series solution, $T = 0.1$

Series Solution: (repeated for convenience)

| $\theta$ | 1 | 2 | 3 | | 21 | 22 | 23 | $T$ |
|---|---|---|---|---|---|---|---|---|
| 15.0 | 2.50711 | 2.88530 | 2.89199 | ... | 2.89171 | 2.89171 | 2.89171 | 0.1000 |
| 30.0 | 2.52813 | 2.91557 | 2.92349 | ... | 2.92326 | 2.92326 | 2.92326 | 0.1000 |
| 45.0 | 2.56211 | 2.96397 | 2.97368 | ... | 2.97351 | 2.97351 | 2.97351 | 0.1000 |
| 60.0 | 2.60747 | 3.02762 | 3.03939 | ... | 3.03924 | 3.03924 | 3.03924 | 0.1000 |
| 75.0 | 2.66201 | 3.10273 | 3.11655 | ... | 3.11641 | 3.11641 | 3.11641 | 0.1000 |

Gauss Method:

| $\theta$ | 1 | 2 | 3 | | 21 | 22 | 23 | $T$ |
|---|---|---|---|---|---|---|---|---|
| 15.0 | 2.67394 | 2.89500 | 2.89166 | ... | 2.89171 | 2.89171 | 2.89171 | 0.1000 |
| 30.0 | 2.16686 | 2.98185 | 2.92002 | ... | 2.92326 | 2.92326 | 2.92326 | 0.1000 |
| 45.0 | 1.57486 | 3.35599 | 2.92979 | ... | 2.97351 | 2.97351 | 2.97351 | 0.1000 |
| 60.0 | 1.04310 | 5.36176 | 3.03939 | ... | 3.03924 | 3.03924 | 3.03924 | 0.1000 |

The Gauss method failed to converge for $\theta \geq 75°$.

*Particular Cases*

In order to numerically integrate the equations of motion for an n-body orbit problem, an initial estimate of a trajectory must be provided. Generally, the two body solution is used to obtain this trajectory information when one gravitational source is predominant over the others. The two body series solution presented in the analytical development section was used to provide the initial trajectories for two multiple body problems. The first case resulted in a hyperbolic transfer, and the second produced an elliptic transfer.

The following values may be considered to be given data for the two-body problem. The position coordinates are given in Astronomical Units, ($AU$) where one $AU$ is the mean distance from the Sun to the Earth. The time unit used is the day, ($d$) where one day equals 24 hours. Angular values are presented in degrees. The gravitational constant $k = \sqrt{\mu}$. The first case was as follows:

$$\begin{aligned}
x1 &= 0.46918988885509\, AU \\
y1 &= -0.77383205171227\, AU \\
z1 &= -0.01964834734771\, AU \\
x2 &= 1.31776281141600\, AU \\
y2 &= -0.41736193703330\, AU \\
z2 &= 0.02991885008669\, AU \\
k &= 0.01720209895000\, AU^{(3/2)}/d \\
t &= 40.000\, d \\
r1 &= 0.90517470889026\, AU \\
r2 &= 1.38260079242914\, AU \\
\theta &= 41.26824^{o} \\
T &= -0.0059691678669
\end{aligned}$$

From the given data, the $B_i$ coefficients of equation (23) may be calculated. A partial sum is formed of the first $i$ terms, then the result is multiplied by the quantity $(s/2)$, where $s$ is the semi-perimeter. This product gives the semi-major axis. The following output shows the value of semi-major axis obtained by using the first $i$ terms in the series. The first number is the index $i$, and the second number is the semi-major axis in $(AU)$.

| | |
|---|---|
| 1 | −49.2301806515044904 |
| 2 | −48.7672453986109379 |
| 3 | −48.7679313499686148 |
| 4 | −48.7679320992466774 |
| 5 | −48.7679321023198326 |
| 6 | −48.7679321023313657 |
| 7 | −48.7679321023314029 |
| 8 | −48.7679321023314030 |
| 9 | −48.7679321023314030 |

Once the semi-major axis has been found, the orbit is completely determined since two positions on the orbit are known from the given data. Any of the remaining classical orbital elements may be calculated as well as position and velocity vectors for any point in the orbit. In the multibody problems it was desired to know the components of the velocity vector at position $\vec{r_1}$, so they are provided for reference. These components were used as the initial value in a boundary value problem.

$$
\begin{aligned}
\dot{x} &= 2.52566922684011E-0002\ AU/d \\
\dot{y} &= 5.02133832285282E-0003\ AU/d \\
\dot{z} &= 1.20446318655231E-0003\ AU/d
\end{aligned}
$$

Following the same format as before, the second case was as follows:

$$
\begin{aligned}
x1 &= 0.50186422427732\,AU \\
y1 &= -0.77640603245208\,AU \\
z1 &= -0.01549685878577\,AU \\
x2 &= 1.37003894998300\,AU \\
y2 &= -0.21022615184980\,AU \\
z2 &= 0.02453126302031\,AU \\
k &= 0.01720209895000\,AU^{(3/2)}/d \\
t &= 54.000\,d \\
r1 &= 0.92461569285281\,AU \\
r2 &= 1.38629129055097\,AU \\
\theta &= 48.43571^{o} \\
T &= 0.1886166547276
\end{aligned}
$$

i, semi-major axis (AU):

| | |
|---|---|
| 1 | 1.58419934014415077 |
| 2 | 2.05984061297696060 |
| 3 | 2.08346900567006944 |
| 4 | 2.08277664547186128 |
| 5 | 2.08286558797827610 |
| 6 | 2.08285434745162601 |
| 7 | 2.08285556050262464 |
| 8 | 2.08285544531016136 |
| 9 | 2.08285545548735275 |
| 10 | 2.08285545459028151 |
| 11 | 2.08285545467360664 |
| 12 | 2.08285545466545692 |
| 13 | 2.08285545466626006 |
| 14 | 2.08285545466618325 |
| 15 | 2.08285545466619034 |
| 16 | 2.08285545466618969 |
| 17 | 2.08285545466618975 |
| 18 | 2.08285545466618975 |
| 19 | 2.08285545466618975 |
| 20 | 2.08285545466618975 |

initial velocity components:

$$\begin{aligned}
\dot{x} &= 2.14961598862402E-0002\ AU/d \\
\dot{y} &= 5.95134600445128E-0003\ AU/d \\
\dot{z} &= 7.08698265474608E-0004\ AU/d
\end{aligned}$$

## IV. *Conclusions and Recommendations*

### *Summary and Conclusions*

The two body series solution presented in the analytical development section appears to be well behaved within the following limits: $-1 \leq T \leq T_{me}$, $3 \leq i \leq 23$. Note when $T = 0$, the orbit is parabolic, and the semi-major axis is infinite. However, the parabolic orbit is uniquely defined for a given problem goemetry.

The series solution also appears to be relatively insensitive to the transfer angle, $\theta$. Presumably this may be explained by noting Lambert's Time Function, from which the series was derived, is also relatively insensitive to the transfer angle.

The series may be used for any two body orbit flight time from hyperbolic transfers up to the minimum energy elliptic transfer. The series should not be used for flight times exceeding the minimum energy transfer time, because the form of Lambert's Time Function used in the derivation does not apply to such orbits. For flight times greater than $t_{me}$, Lambert's Time Function includes a constant term related to the orbital period, which is the time required to make one complete orbital revolution. The presence of this term makes it difficult to express Lambert's Time Function as a power series.

Because the series may be used for both hyperbolic and elliptic orbits, it avoids the necessity of programming for separate cases as is usually done in the direct application of Lambert's Time Function.

The greatest advantage gained by using the series solution for semi-major axis over other methods is that no initial value is required, because no root-finding technique is necessary. Given the flight time and orbital geometry, it is possible to substitute the appropriate values in to the series definition, then evaluate the series to obtain the semi-major axis of the orbit.

*Recommendation for Further Study*

The form of Lambert's Time Function for orbital transfers greater than the minimum energy transfer time does not lend itself to series representation. If a series representation were to be found, then the inversion/reversion techniques presented in the analytical development section may prove useful in solving such a series.

It may be possible to use Asymptotic Matching (5: p. 270-279), to find the series that describes greater that minimum energy orbits. By allowing the value of semi-major axis to approach infinity in Lambert's Time Function, one may obtain a relationship between flight time and semi-major axis for very long transfer times. This may be matched to the solution for transfer times approaching the minimum energy time. If the resulting expression could be expressed as a series, the reversion/inversion techniques again may be useful in solving the series.

# Appendix A. *Example Coefficients*

The first six coefficients are given:

$$
\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2/A_1 \\
B_3 &= (A_1A_3 - A_2^2)/A_1^3 \\
B_4 &= (A_1^2A_4 - 3A_1A_2A_3 + 2A_2^3)/A_1^5 \\
B_5 &= (A_1^3A_5 - 4A_1^2A_2A_4 - 2A_1^2A_3^2 + 10A_1A_2^2A_3 - 5A_2^4)/A_1^7 \\
B_6 &= (A_1^4A_6 - 5A_1^3A_2A_5 + 15A_1^2A_2^2A_4 - 5A_1^3A_3A_4 + 15A_1^2A_2A_3^2 \\
&\quad -35A_1A_2^3A_3 + 14A_2^5)/A_1^9 \\
B_7 &= (A_1^5A_7 - 6A_1^4A_2A_6 + 21A_1^3A_2^2A_5 - 6A_1^4A_3A_5 - 3A_1^4A_4^2 + 42A_1^3A_2A_3A_4 \\
&\quad -56A_1^2A_2^3A_4 + 7A_1^3A_3^3 - 84A_1^2A_2^2A_3^2 + 126A_1A_2^4A_3 - 42A_2^6)/A_1^{11}
\end{aligned}
$$

It is not recommended that these equations be programmed directly due to their complexity. To avoid coding errors, it would be more efficient to program the recursive relationships developed in the analytical development section. The recursive algorithm allows the calculation of terms of much higher order than the first seven without explicit coding. A sample FORTRAN listing is included in Appendix C that includes the necessary algorithms.

## Appendix B. *Accuracy Plots for Various Transfer Angles*

Plots of accuracy vs. T and number of terms are given for the following values of transfer angle: $\theta = 90^o, 180^o, 270^o, 360^o$. The vertical axis represents the number of significant figures of accuracy in the calculated value of semi-major axis. In general, the number of significant figures increases with the number of terms in the series. The increase is largest in the neighborhood of the Parabolic Transfer Time which corresponds to $T = 0$.

An example point is shown on figure 3, where the indicated point has coordinates $T = -3$, $i = 8$, and significant figures = 7.

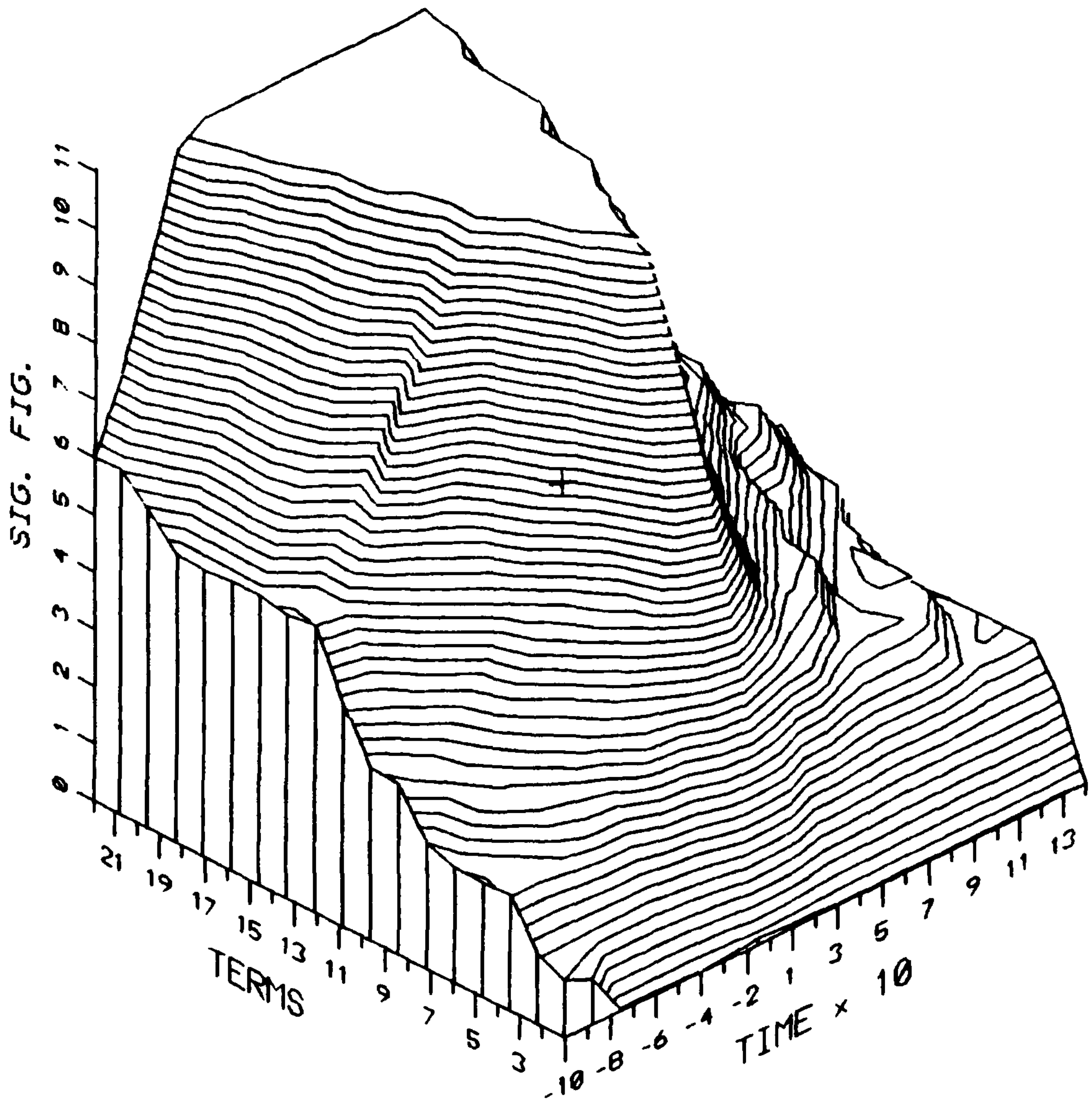


Figure 3. Accuracy plot for 90 degree transfer angle.

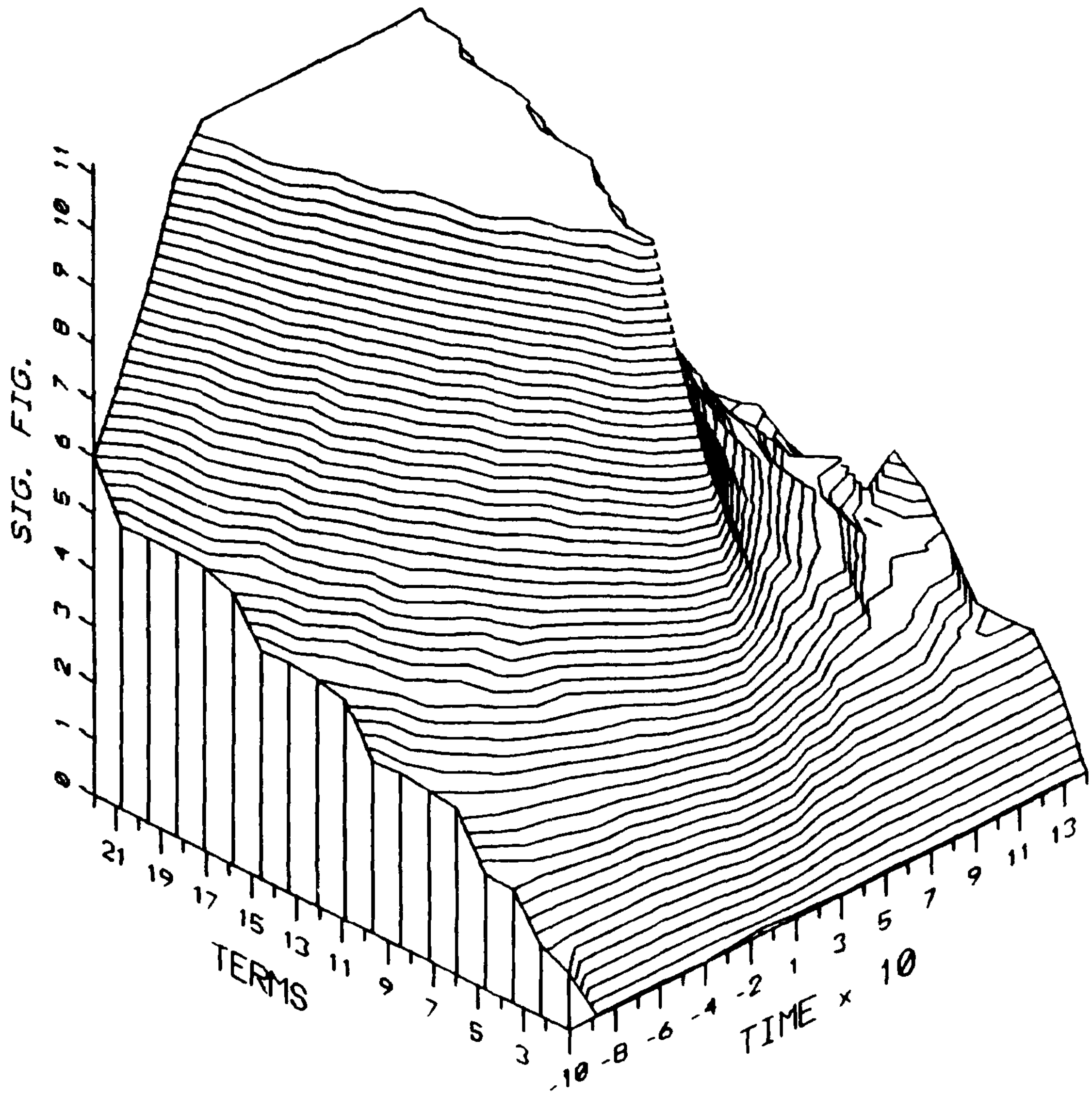


Figure 4. Accuracy plot for 180 degree transfer angle.

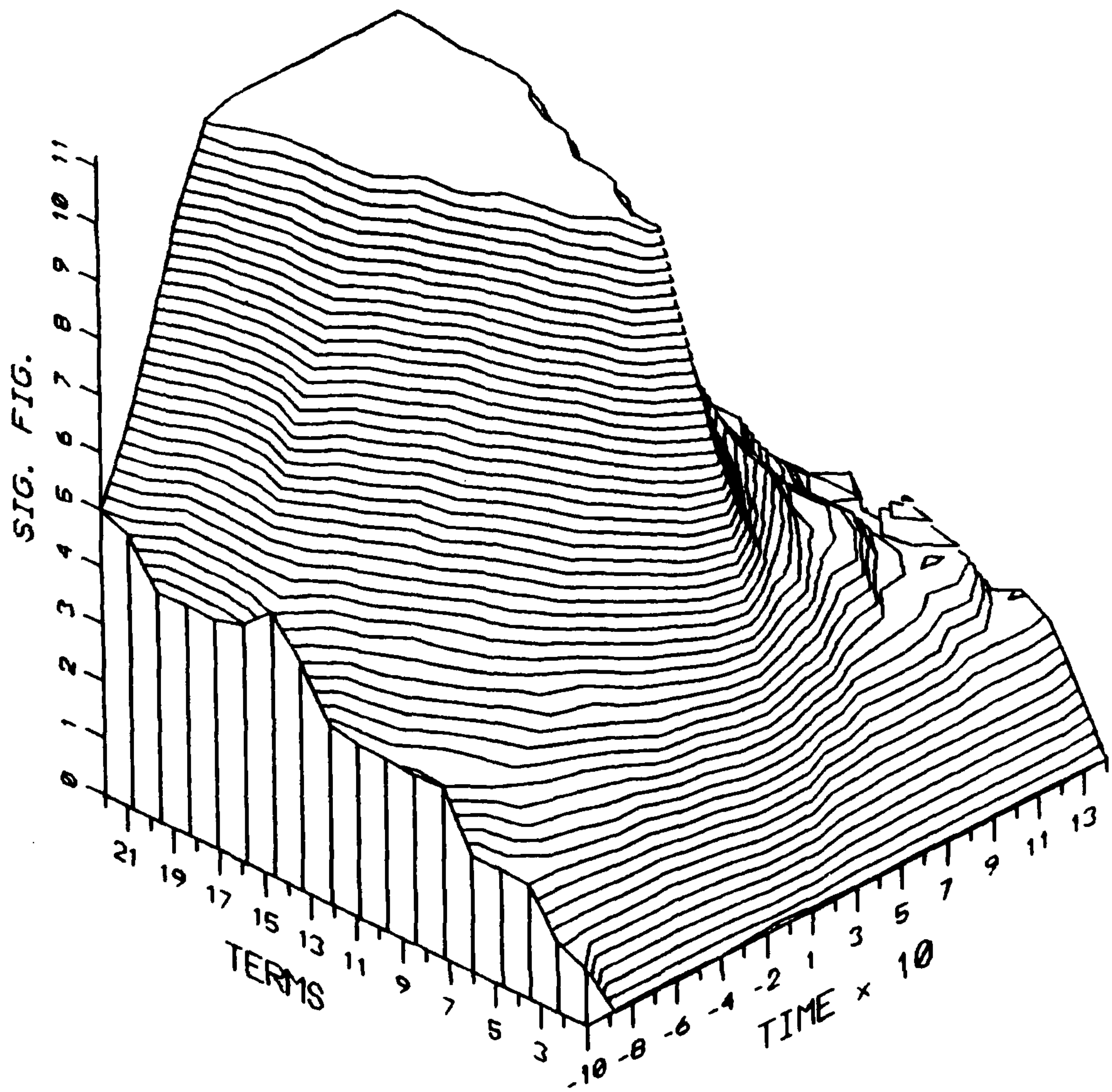


Figure 5. Accuracy plot for 270 degree transfer angle.

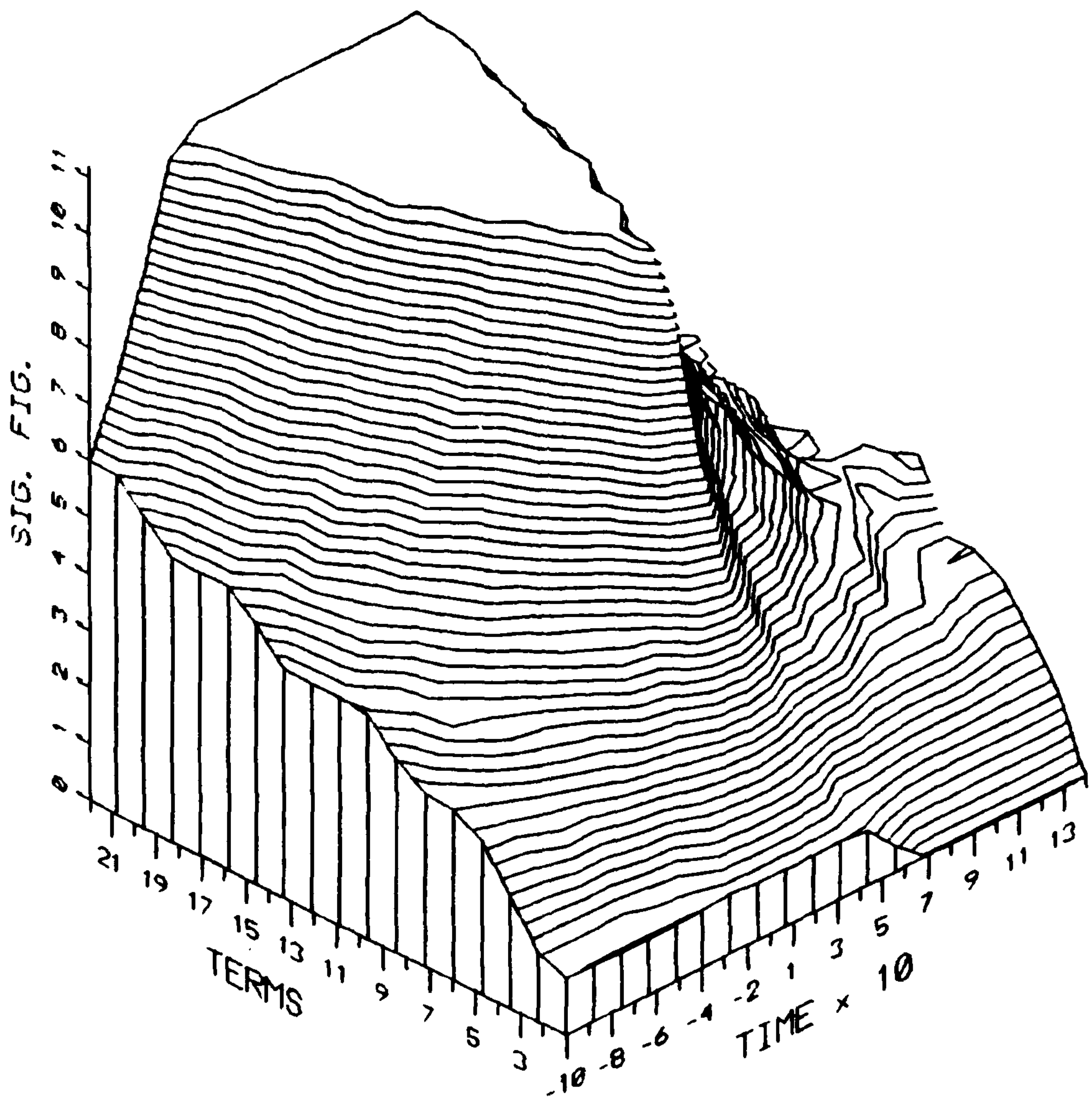


Figure 6. Accuracy plot for 360 degree transfer angle.

## Appendix C. *FORTRAN listing of Algorithm*

This is the FORTRAN listing of the algorithm. The program requires two position vectors, a gravitational constant, and a flight time, and will produce the corresponding value of semi-major axis.

```
      PROGRAM ELAH9
C
C     THIS VERSION WILL HANDLE HYPERBOLIC AND ELLIPTIC ORBITS
C
      DOUBLE PRECISION Q(78,78) ,  A(78) , B(78)
      DOUBLE PRECISION X, XB, S
      DOUBLE PRECISION TOF , TIME , PARATIME , SEMIMAJORAXIS
      DOUBLE PRECISION X1 , Y1 , Z1 , X2 , Y2 , Z2 , K
      DOUBLE PRECISION R1 , R2 , DOT , ARG , BETA
      DOUBLE PRECISION CHORD , SEMIPERIMETER
      DOUBLE PRECISION K1 , K2 , K3
      DOUBLE PRECISION ARG1 , ARG2 , C1 , C2 , C3 , TC
      DOUBLE PRECISION RV1 , RV2 , RV3 , AL , BE , EA , F , G
      DOUBLE PRECISION V1 , V2 , V3 , II

      INTEGER          I , J , M , N , FLAG , FLAG2
      INTEGER          ORDER , AA , BB , P

      COMMON /ELAHCOM/ Q, A, B, X, XB, S, TOF, TIME, PARATIME,
     +                 SEMIMAJORAXIS, X1, Y1, Z1, X2, Y2, Z2, K,
     +                 R1, R2, DOT, ARG, BETA, CHORD, SEMIPERIMETER,
     +                 K1, K2, K3, ARG1, ARG2, C1, C2, C3, TC, RV1,
     +                 RV2, RV3, AL, BE, EA, F, G, V1, V2, V3, II,
     +                 I, J, M, N, FLAG, FLAG2, ORDER, AA, BB, P

      DOUBLE PRECISION PI
      PARAMETER (PI = 3.141592653589D0)

C     DEFINE THE FUNCTIONS FACT AND POCH

      DOUBLE PRECISION FACT, POCH, TEMP

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C     BEGINNING OF MAIN PROGRAM
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

C     GEOMETRY

        x1 =   0.50186422427732D0
        y1 =  -0.77640603245208D0
        z1 =  -0.01549685878577D0

        x2 =   1.37003894998300D0
        y2 =  -0.21022615184980D0
        z2 =   0.02453126302031D0

C       x1 =   0.46918988885509D0
C       y1 =  -0.77383205171227D0
C       z1 =  -0.01964834734771D0

C       x2 =   1.31776281141600D0
C       y2 =  -0.41736193703330D0
C       z2 =   0.02991885008669D0

C       SOLAR GRAVITATIONAL CONSTANT ( AU^3/2 / DAY )

        K  =   0.01720209895000D0
```

Figure 7. FORTRAN listing of algorithm

```
      R1  = DSQRT( X1 * X1 + Y1 * Y1 + Z1 * Z1 )
      R2  = DSQRT( X2 * X2 + Y2 * Y2 + Z2 * Z2 )
      DOT = (XI * X2) + (Y1 * Y2) + (Z1 * Z2)

      BETA = DACOS( DOT / (R1 * R2) )

      IF (BETA .LT. PI) THEN
         FLAG = -1
      ELSE
         FLAG = 1
      ENDIF

      CHORD = DSQRT(R1 * R1 + R2 * R2 - 2.0D0 * R1 * R2 * DCOS(BETA))
      SEMIPERIMETER = (R1 + R2 + CHORD) / 2.0D0

      PARATIME = (DSQRT(2.0D0) / K) * (SEMIPERIMETER**1.5D0 +
     +           DBLE(FLAG) * (SEMIPERIMETER - CHORD)**1.5D0) / 3.0D0

      K1 = DSQRT(SEMIPERIMETER**3.0D0 / 2.0D0) / K
      K2 = DBLE(FLAG) * DSQRT((SEMIPERIMETER-CHORD)**3.0D0 / 2.0D0) /K
      K3 = (SEMIPERIMETER - CHORD) / SEMIPERIMETER

C     INITIALIZE MATRIX

      ORDER = 23

      DO 20 I = 1,ORDER
         DO 10 J = 1,ORDER
            Q(I,J) = 0.0D0
10       CONTINUE
         A(I) = 0.0D0
         B(I) = 0.0D0
20    CONTINUE

C     CALCULATE COEFFICIENTS A_I OF ORIGINAL SERIES


      DO 30 I = 1,ORDER
         TEMP = DBLE(I)
         A(I) = POCH(1.5D0,TEMP) * POCH(0.5D0,TEMP) / POCH(2.5D0,TEMP)
         A(I) = A(I) * ((1.0D0 + DBLE(FLAG) * K3**(TEMP + 1.5D0)) /
     +                 (1.0D0 + DBLE(FLAG) * K3**1.5D0))  / FACT(TEMP)
30    CONTINUE

C     FORM Q MATRIX

      Q(1,1) = 1.0D0 / A(1)

      DO 70 I = 2,ORDER
         DO 50 J = 2,ORDER
            IF (I .GE. J) THEN
               DO 40 M = 1,(I - 1)
                  Q(I,J) = Q(I,J) + Q(I-M,1) * Q(M,(J-1))
40             CONTINUE
            ENDIF
50       CONTINUE
         DO 60 N = 2,I
            Q(I,1) = Q(I,1) + A(N) * Q(I,N)
60       CONTINUE
```

Figure 8. FORTRAN listing of algorithm (continued)

```
         Q(I,1) = Q(I,1) / (-A(1))
70    CONTINUE

C     USE A AND Q TO GET FINAL COEFFICIENTS

      B(1) = A(1)

      DO 90 I = 2,ORDER
         DO 80 M = 2,I
            B(I) = B(I) + A(M) * Q((I-1),(M-1))
80       CONTINUE
90    CONTINUE

C     USE COEFFICIENTS WITH DIFFERENT TIMES


C        INITIALIZE

            X = 0.0D0
            XB = 0.0D0
            S = 0.0D0

            TOF = 54.0D0
            TIME = TOF / PARATIME - 1.0D0

         DO 300 I = 1,ORDER

               XB = B(I) * TIME**(I-2)
               X = X + XB
               S = X * SEMIPERIMETER / 2.0D0
               PRINT *, I, S

300      CONTINUE

      STOP
      END

      DOUBLE PRECISION FUNCTION FACT(U)

C     CALCULATES FACTORIALS

      DOUBLE PRECISION U, PROD, TEMP

CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C     BEGINNING OF FUNCTION FACT
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC

      IF (U .EQ. 0.0D0) THEN
         FACT = 1.0D0
      ELSE
         PROD = U
         TEMP = U

10       CONTINUE

         IF (TEMP .GT. 1.0D0) THEN
            TEMP = TEMP - 1.0D0
            PROD = PROD * TEMP
```

Figure 9. FORTRAN listing of algorithm (continued)

```
            GOTO 10
         ENDIF
         FACT = PROD
      ENDIF

      END

      DOUBLE PRECISION FUNCTION POCH(U,V)

C     CALCULATES POCHHAMMER SYMBOLS

      DOUBLE PRECISION U,V,PROD,TEMP

cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
C
C     BEGINNING OF FUNCTION POCH
C
cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc

      PROD = 1.0D0
      TEMP = V

10    CONTINUE

      IF (TEMP .GE. 1.0D0) THEN
         TEMP = TEMP - 1.0D0
         PROD = PROD * (U + TEMP)
         GOTO 10
      ENDIF

      POCH = PROD

      END
```

Figure 10. FORTRAN listing of algorithm (continued)

## *Bibliography*


1. Andrews, Larry C., *Special Functions for Engineers and Applied Mathematicians.* New York: Macmillan Publishing Company, 1985.
2. Arfken, George, *Mathematical Methods for Physicists.* Orlando, Florida: Academic Press Inc., 1985.
3. Battin, Richard H., *Astronautical Guidance.* New York: McGraw-Hill Book Company, 1964.
4. Escobal, Pedro R., *Methods of Orbit Determination.* Malabar, Florida: Robert E. Krieger Publishing Company, 1965.
5. Nayfeh, Ali H., *Introduction to Perturbation Techniques.* New York: John Wiley and Sons, 1981.

## *Vita*

Captain James Dana Thorne [redacted] He graduated from Grove City High School, Grove City PA in 1980, and enrolled in the astronautical engineering program at Purdue University, West Lafayette, IN. After completing a Bachelor of Science degree , he was commissioned through ROTC on 12 May, 1984. During an assignment to the Air Force Satellite Control Facility in Sunnyvale, CA, he met [redacted] who was employed by Lockheed Missiles and Space Corporation as a scientific programmer. They were married 27 September 1986, and relocated to Wright-Patterson AFB in April, 1988 for an AFIT master's program in astronautical engineering.

[redacted]